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Fully Symbolic Model Checking of Timed Systems using Difference Decision Diagrams

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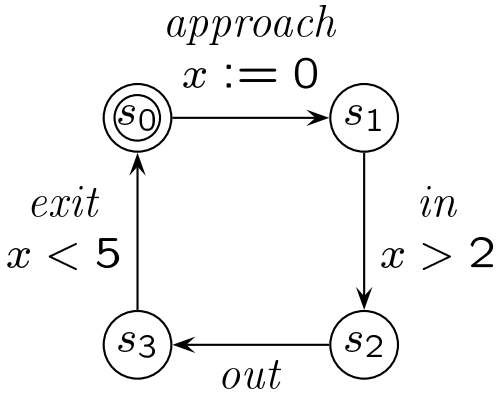
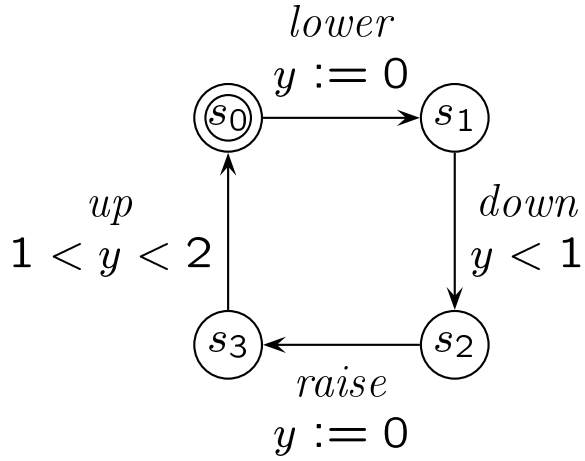
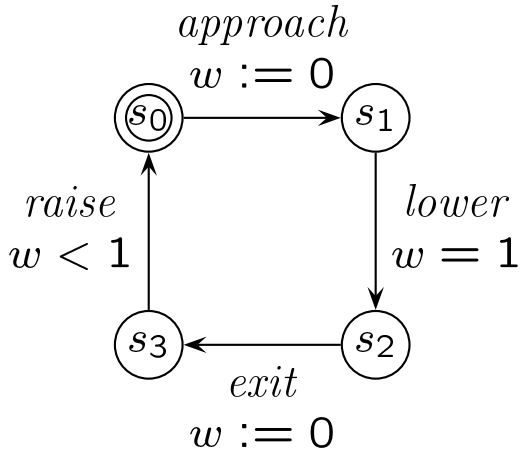
Example: Gate controller at a railroad crossing [Alur & Dill 91].



Train



Gate



Controller

Safety When the train is inside the gate, the gate should be closed.

Liveness The gate should never be closed for more than 10 minutes.

Current Approaches

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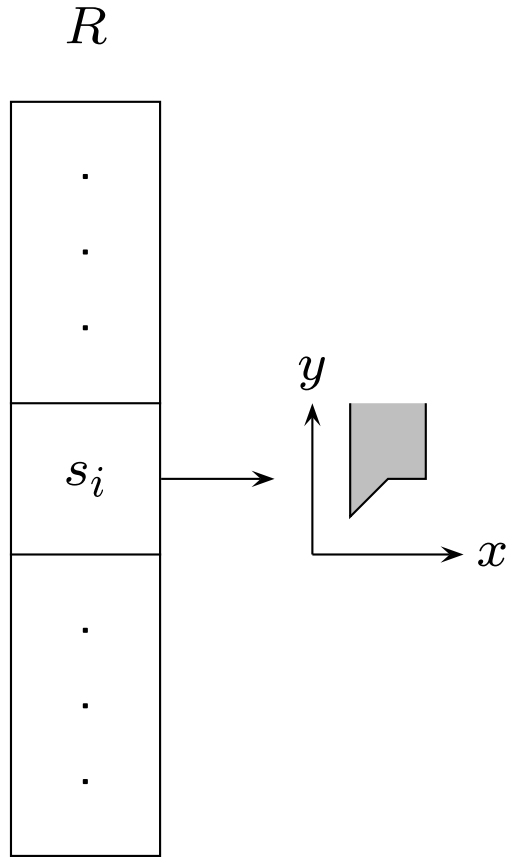
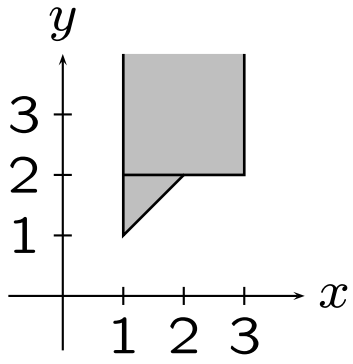
$Q \leftarrow \{(s_0, V_0)\}$
while $Q \neq \emptyset$ **do**
  Remove some (s, V) from Q
  $S \leftarrow \text{Next}(s, V)$
  for each $(s_i, V_i) \in S$ **do**
    if $V_i \not\subseteq R[s_i]$ **then**
      Add (s_i, V_i) to Q
      $R[s_i] \leftarrow R[s_i] \cup V_i$



DBM-based representation [Uppaal]



$$\begin{array}{c} \\ 0 \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccc} 0 & x & y \end{array} \\ \begin{bmatrix} 0 & 3 & \infty \\ -1 & 0 & \infty \\ -2 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ 0 \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccc} 0 & x & y \end{array} \\ \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \end{array}$$

Disadvantages:

1. The number of DBMs for representing $R[s_i]$ can become very large.
2. There is no sharing of DBMs between different states.
3. Each discrete state is enumerated explicitly.

A Symbolic Approach

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Represent a set of states as a formula ϕ.

$\phi \leftarrow \phi_0$
$R \leftarrow \phi$
while $\phi \not\equiv \mathit{false}$ **do**
  $\phi \leftarrow \mathsf{Next}(\phi) \wedge \neg R$
  $R \leftarrow R \vee \phi$

$$R \;=\; \Big((s_0 \vee s_1) \wedge 0 \leq x - y < 5\Big) \vee \cdots$$

We need : 1. A logic to express formulas in.
2. Next(ϕ).
3. A data structure for representing formulas.

Outline

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- Timed guarded commands
- Symbolic reachability analysis
- Difference decision diagrams
- Experimental results

Timed Guarded Commands

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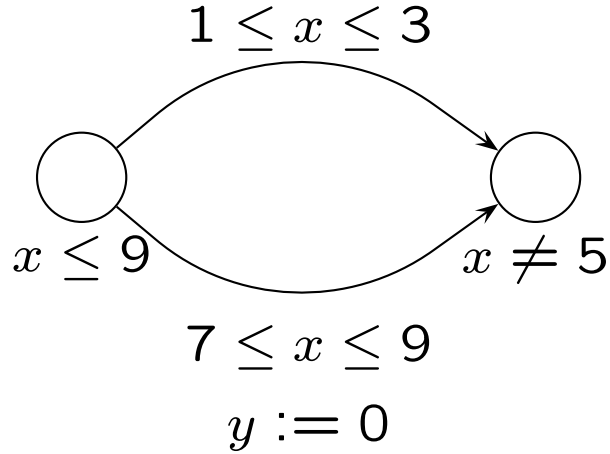
$$
\begin{aligned}
G &= (\{b\}, \{x, y\}, T, I)\,, \\
T &= \left\{ \begin{array}{ll} b \wedge (1 \leq x \leq 3) \rightarrow b & := \mathsf{false} \\ b \wedge (7 \leq x \leq 9) \rightarrow b, y & := \mathsf{false}, 0 \end{array} \right\}, \\
I &= \Big(b \Rightarrow (x \leq 9)\Big) \wedge \Big(\neg b \Rightarrow (x \neq 5)\Big)\,.
\end{aligned}
$$



Transitional Semantics

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A program state: $s = [x \mapsto 8.0,\ y \mapsto 0.0,\ b \mapsto \mathit{false}]$.

- **Discrete transition**

  Firing $x < 5.0 \ \rightarrow \ b, x := \mathit{true}, 0.0$:

  $$s' = [x \mapsto 0.0,\ y \mapsto 0.0,\ b \mapsto \mathit{true}]$$

- **Timed transition**

  Advancing time by 2.5:

  $$s' = [x \mapsto 10.5,\ y \mapsto 2.5,\ b \mapsto \mathit{false}]$$

The Logic

The logic to express formulas in :

$$\phi \ ::= \ x - y \leq d \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \exists x.\phi \, .$$

We introduce the variable z representing "zero" or "current time".

Replace $x \leq d$ by $x - z \leq d$. Replace $x := d$ by $x := z + d$.

Example:

$$x \leq 5 \ \rightarrow \ y := 1 \qquad \text{becomes} \qquad x - z \leq 5 \ \rightarrow \ y := z + 1$$

Discrete Transitions

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Firing $g \to \vec{v} := \vec{d}$:

$$\frac{s \models g \quad s[\vec{v} := \vec{d}] \models I}{s \to s[\vec{v} := \vec{d}]}$$

$$\mathsf{Next}_{\mathsf{discrete}}(\phi,\ g \to \vec{v} := \vec{d}) \ = \ (\phi \wedge g)[\vec{v} := \vec{d}] \wedge I$$

Advancing time using the z-variable

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Advancing all clocks by δ:

$$\phi[\vec{c} := \vec{c} + \delta] \;=\; \phi[z := z - \delta] \;=\; \Big(\exists z.(\phi \wedge z' = z - \delta)\Big)[z/z']$$

Timed Transitions

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Advance time by δ :

$$\frac{\delta \geq 0 \quad \forall \delta'.0 \leq \delta' \leq \delta : s[\vec{c} := \vec{c} + \delta'] \models I}{s \xrightarrow{\delta} s[\vec{c} := \vec{c} + \delta]}$$

Predicate stating whether z' is a legal zero-point :

$$P_{\mathsf{next}} \;=\; z' \leq z \;\wedge\; \forall z''.(z' \leq z'' \leq z \Rightarrow I[z''/z])$$

$$\mathsf{Next}_{\mathsf{timed}}(\phi, \delta) \;=\; \Big(\exists z.(\phi \;\wedge\; z' = z - \delta \;\wedge\; P_{\mathsf{next}})\Big)[z/z']$$

$$\mathsf{Next}_{\mathsf{timed}}(\phi) \;=\; \bigvee_{\delta \in \mathbb{R}} \mathsf{Next}_{\mathsf{timed}}(\phi, \delta) \;=\; \Big(\exists z.(\phi \wedge P_{\mathsf{next}})\Big)[z/z']$$

Example

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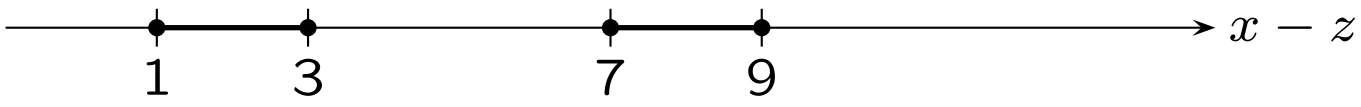
Consider the formula

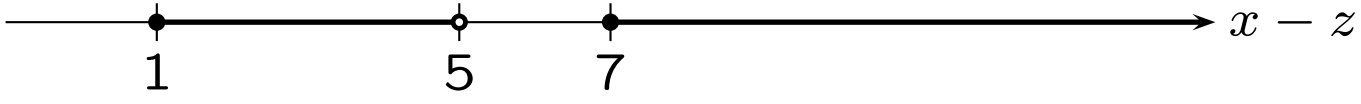
$$\phi \;=\; (1 \leq x - z \leq 3) \vee (7 \leq x - z \leq 9)$$



and the state-invariant $I \;=\; x - z \neq 5$.

Advancing time from ϕ:

$$\mathsf{Next}_{\mathsf{timed}}(\phi) \;=\; (1 \leq x - z < 5) \vee (7 \leq x - z)$$



The 3 Ingredients

- ✓ A logic
- ✓ Next(ϕ)
- ☛ A data structure

Difference Decision Diagrams

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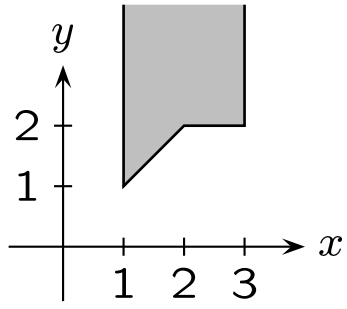
Implicit representation of:

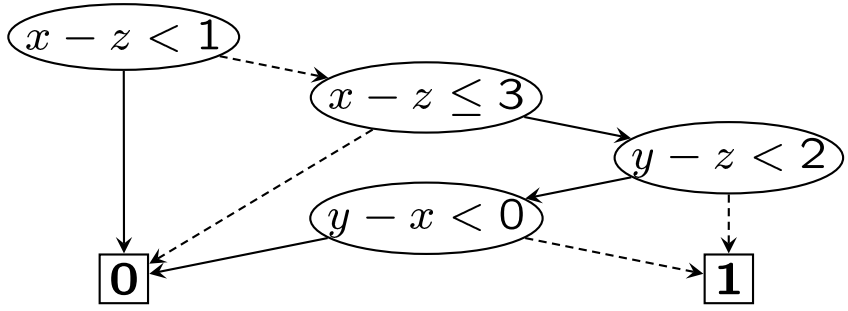
$$\phi \ ::= \ x - y \leq d \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \exists x.\phi\,.$$

Example

$$\phi \ = \ 1 \leq x - z \leq 3 \ \wedge \ (y - z \geq 2 \ \vee \ y - x \geq 0)$$

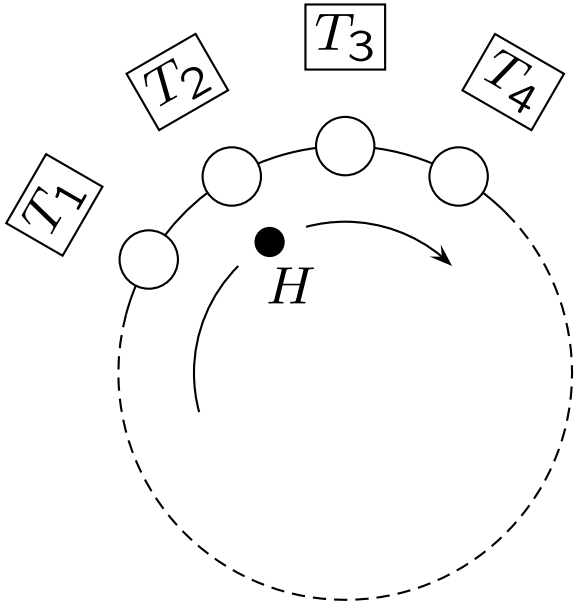


(x, y)-plot for $z = 0$



Difference decision diagram

Milner's Scheduler [Milner 89]



$$
\begin{array}{lcll}
c_i \wedge \neg t_i & \rightarrow & H, T_i, t_i, c_i, h_i & := 0, 0, \mathsf{T}, \mathsf{F}, \mathsf{T} \\
h_i \wedge H^l \leq H & \rightarrow & c_{(i \bmod N)+1}, h_i & := \mathsf{T}, \mathsf{F} \\
t_i \wedge T^l \leq T_i & \rightarrow & t_i & := \mathsf{F}
\end{array}
$$

$$I = \bigwedge_{i=1}^{N} (h_i \Rightarrow H \leq H^u) \wedge (t_i \Rightarrow T_i \leq T^u)$$

Experimental Results

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N	Kronos	Uppaal	DDD
4	0.2	0.1	0.1
5	0.7	0.2	0.1
6	22.6	0.6	0.1
7	339.2	2.3	0.1
8	–	9.0	0.2
9	–	35.0	0.2
10	–	138.4	0.2
11	–	529.8	0.2
12	–	2560.7	0.3
16	–	–	0.5
32	–	–	2.2
64	–	–	15.9
128	–	–	123.3
256	–	–	1104.8

(a) 1 clock.

N	Kronos	Uppaal	DDD
4	0.4	0.2	0.2
5	2.4	1.7	0.3
6	24.2	17.6	0.5
7	346.6	201.7	0.5
8	–	2460.2	0.6
16	–	–	1.5
32	–	–	5.7
64	–	–	31.7
128	–	–	217.3

(b) $N + 1$ clocks.

Conclusion

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- Symbolic model checking of timed systems requires a data structure and a decision procedure for the logic :

$$\phi \ ::= \ x - y \leq d \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \exists x.\phi \, .$$

- Advancing time amounts to existential quantification of the z-variable.

- TCTL formulas can be verified by letting time go backwards.